

Semi-classical beam cooling in an intense laser pulse

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We present a novel technique for studying the evolution of a particle distribution using single particle dynamics such that the distribution can be accurately reconstructed using fewer particles than existing approaches. To demonstrate this, the Landau–Lifshitz description of radiation reaction is adapted into a semi-classical model, for which the Vlasov equation is intractable. Collision between an energetic electron bunch and high-intensity laser pulses are then compared using the two theories. Reduction in beam cooling is observed for the semi-classical case.

The emergence over the next few years of a new generation of ultra-high power laser facilities, spearheaded by the Extreme Light Infrastructure [1] (ELI), represents a major advance in the possibilities afforded by laser technology. As well as important practical applications, these facilities will for the first time allow us to probe qualitatively new physical regimes. One of the first effects to be explored will be radiation reaction.

Radiation reaction—the recoil force on an electron due to its emission of radiation—remains a contentious area of physics after more than a century of investigation. The standard equation describing radiation reaction (the so-called LAD equation, after its progenitors Lorentz, Abraham, and Dirac [2–4]) for a particle [5] of mass m and charge q in an electromagnetic field F reads

$$\ddot{x}^a = -\frac{q}{m}F^a{}_b\dot{x}^b + \tau\Delta^a{}_b\ddot{x}^b. \quad (1)$$

Here $\tau := q^2/6\pi m$ is the ‘characteristic time’ of the particle ($\simeq 6 \times 10^{-24}$ s for an electron); $\Delta^a{}_b := \delta^a_b + \dot{x}^a\dot{x}_b$ is the \dot{x} -orthogonal projection; and an overdot denotes differentiation with respect to proper time. Indices are raised and lowered with the metric tensor $\eta = \text{diag}(-1, 1, 1, 1)$, and repeated indices are summed from 0 to 3. We work in Heaviside-Lorentz units with $c = 1$.

Despite numerous independent derivations of equation (1) [4, 6, 7], it is subject to numerous difficulties; see the recent review [8] for an account of these problems, and some of the proposed solutions. The most widely used alternative is that introduced by Landau and Lifshitz [9], by treating the self-force as a small perturbation about the applied force:

$$\ddot{x}^a = -\frac{q}{m}F^a{}_b\dot{x}^b - \tau\frac{q}{m}\left(\dot{F}^a{}_b\dot{x}^b - \frac{q}{m}\Delta^a{}_bF^{bc}F_{cd}\dot{x}^d\right). \quad (2)$$

It is often claimed that (2) is valid provided only that quantum effects can be ignored, and though a rigorous demonstration remains elusive there is mounting evidence that this is indeed the case [10].

Under the conditions expected at ELI, the caveat ‘provided that quantum effects can be ignored’ is pertinent. Quantum effects are typically negligible if the electric field observed by the particle is much less than the

Sauter-Schwinger field [11, 12] typical of QED processes,

$$\chi := \frac{e\hbar}{m_e^2}\sqrt{F^{ab}F_{ac}\dot{x}_b\dot{x}^c} \ll 1. \quad (3)$$

For 1 GeV electrons in a laser pulse of intensity 10^{22} W/cm² (parameters typical of ELI), $\chi \sim 0.8$ and quantum effects cannot be ignored. A complete QED treatment of radiation reaction is problematic to define but, provided χ remains less than about unity, a semi-classical modification to (2) should be valid [13].

It is generally accepted that (classical or quantum) radiation reaction effects will be more readily observed in the behaviour of particles than in the radiation they emit [14, 15]. As such, it is important to be able to accurately determine the distribution of a bunch of particles evolving according to (2) or its semi-classical extension. Usually this would involve solving a Vlasov equation or following the evolution of very large numbers of particles, either of which is computationally very intensive.

An important difference between the classical and quantum picture of radiation emission by a charged particle can be seen in the radiation spectrum. Classically, a charged particle can radiate at all frequencies. However, according to the quantum picture, the energy of the emitted photons is limited by the energy of the particle. This suppresses emission at high frequencies, and introduces a cutoff to the radiation spectrum [13]. As such, it is expected that the effects of radiation reaction are actually over estimated by classical theories in regimes where quantum effects become important [16].

In order to account for this reduction of radiation reaction effects in the Landau–Lifshitz equation of motion, following Kirk, Bell and Arka [17] we scale the radiation reaction force (the term in parentheses) in (2) by a function $g(\chi)$. The full expression for $g(\chi)$ involves a non-trivial integral over Bessel functions of the second kind. Instead, a fit to data obtained from this function was found by Thomas *et al.* [14] to be

$$g(\chi) = (3.7\chi^3 + 31\chi^2 + 12\chi + 1)^{-4/9}, \quad (4)$$

and it is this model which we adopt here. It can be clearly seen that, as $\chi \rightarrow 0$, we have $g(\chi) \rightarrow 1$, therefore

recovering the classical equation of motion. This model essentially reduces to a rescaling of the characteristic time of the electron, $\tau \rightarrow g(\chi)\tau$. For $\chi \gtrsim 1$, the stochasticity of quantum emission becomes important, and the semi-classical model is no longer applicable [18].

The motion of a single charged particle colliding with a laser pulse in the radiation reaction regime has been well studied [19–21]. Instead of having to solve a Vlasov-type equation on the phase space, we propose a novel numerical method which allows for the dynamics of a particle *distribution* to be explored using single-particle equations of motion such that the distribution can be cleanly and efficiently reconstructed. While this approach is quite general and could be used for a variety of systems, we here consider a distribution of particles subject to equation (2) and its semi-classical extension, without particle-particle interactions [22]. The Vlasov equation for the latter system has no analytical solution, and would require significant computing resources to solve numerically.

Assuming that the laser pulse can be approximated by a plane wave with compact longitudinal support, any spatial spread in the particle distribution would only define the moment in time when each particular particle enters the pulse. For simplicity, we therefore take all particles to originate from the same point in space. This is reasonable as we are primarily interested in the momentum distribution. This also allows us to consider the particle distribution as a function of the phase $\phi = \omega t - \mathbf{k} \cdot \mathbf{x}$.

Since our pulse is modelled by a plane wave and we focus on the longitudinal properties of the distribution, we consider the momenta to be strongly peaked in the transverse directions. As such, the initial distribution can be taken to be a thermal Maxwell–Boltzmann particle distribution for the (longitudinal) momentum p (in units of mc)

$$f(\phi = 0, p) = \frac{N_P}{\sqrt{2\pi}\theta} \exp\left[-\frac{(p - \bar{p})^2}{2\theta}\right], \quad (5)$$

with the thermal spread $\theta = kT/mc^2$, where k is the Boltzmann constant and N_P is the number of particles. We stress that this initial distribution is chosen for its simplicity; alternative distributions could be used where appropriate.

Typically, one would sample the distribution at random, which would require a large number of particles to accurately represent the distribution. Instead, since the particle number is simply $N_P = \int_{-\infty}^{\infty} dp f(\phi, p)$, we determine the momentum spacing δp between the particles from the initial distribution by truncating the integral, so that the particle number increases by unity in the given momentum interval:

$$1 = \int_{p - \frac{\delta p}{2}}^{p + \frac{\delta p}{2}} dp f(0, p) \simeq f(0, p) \delta p. \quad (6)$$

This leads to a set of $N_p = 2N_c + 1$ initial momenta $V(0) = \{p_i(0)\}$ for $i \in [-N_c, N_c]$, with the p_i generated iteratively from $p_0 = \bar{p}$ and $p_{\pm 1} = \bar{p} \pm 1/f(0, \bar{p})$ using

$$p_i = p_{i-2\xi} + \frac{2\xi}{f(0, p_{i-\xi})} \quad \text{with} \quad \xi = \text{sgn}(i). \quad (7)$$

As the evolution proceeds, this procedure is applied in reverse to reconstruct the distribution. The set of momenta $V(\phi)$ is ordered such that $p_{i+1} \geq p_i$ and used to find $\delta p_i(\phi) = (p_{i+1}(\phi) - p_{i-1}(\phi))/2$. The velocity distribution is then defined to be

$$f(\phi, p_i) := \frac{1}{\delta p_i(\phi)}. \quad (8)$$

Reconstruction of a distribution from a particle sample can be problematic, but in this formalism it becomes quite natural.

It has recently been shown using an analytic solution to the Vlasov equation including radiation reaction according to the Landau–Lifshitz theory [23] that collision with a high-intensity laser pulse leads to a significant contraction of the particle phase space, resulting in a reduction in the relative momentum spread. As previously observed, this beam cooling depends only on the total fluence of the pulse, rather than its duration or peak intensity independently [24].

For the classical Landau–Lifshitz theory, agreement between this Vlasov solution and numerical results obtained using the proposed method for sampling and reconstructing the distribution as discussed above is excellent. However, for the semi-classical extension, the Vlasov equation is no longer tractable. To demonstrate the use of our proposed method in such a case, we consider the importance of quantum effects in the interaction of an electron bunch with a laser pulse modelled by a plane wave.

In order to establish the impact of radiation reaction on the evolution of the particle distribution function, and the consequence of allowing the radiation reaction force to be reduced by the quantum model, we introduce the relative momentum spread and the momentum skewness (calculated from the mean \bar{p} and variance θ):

$$\hat{\sigma}(\phi) = \frac{\sqrt{\theta(\phi)}}{\bar{p}(\phi)} \quad \text{and} \quad S(\phi) = \frac{\langle [p - \bar{p}(\phi)]^3 \rangle}{\theta^{3/2}(\phi)}. \quad (9)$$

The former gives a measure of the beam quality, whereas the latter indicates how symmetric the distribution is about its mean. (See also the Supplemental Material.)

We introduce the (null) wave vector k such that (with our choice of metric) the phase of the pulse may be expressed as $\phi = -k \cdot x = \omega t - \mathbf{k} \cdot \mathbf{x}$. The orthogonal vectors ϵ, λ satisfy $\epsilon^2 = \lambda^2 = 1$ and $k \cdot \epsilon = k \cdot \lambda = \epsilon \cdot \lambda = 0$, and together with k and the null vector ℓ (defined to satisfy $\ell \cdot \epsilon = \ell \cdot \lambda = 0$ and $k \cdot \ell = -1$) form a basis.

For a plane wave with arbitrary polarisation, the electromagnetic field tensor F takes the form

$$\frac{q}{m}F^a_b = a_\epsilon(\phi)(\epsilon^a k_b - k^a \epsilon_b) + a_\lambda(\phi)(\lambda^a k_b - k^a \lambda_b), \quad (10)$$

where the functions $a_{\epsilon,\lambda}(\phi)$ are dimensionless measures of the electric field strength in the ϵ, λ direction. We restrict our attention to a linearly polarised N -cycle pulse, modulated by a \sin^2 -envelope [10], with

$$a(\phi) = \begin{cases} a_0 \sin(\phi) \sin^2(\phi/2N) & \text{for } 0 < \phi < 2\pi N \\ 0 & \text{otherwise} \end{cases}, \quad (11)$$

where a_0 is the dimensionless (peak) intensity parameter (the so-called “normalised vector potential”). This pulse shape offers compact support, allowing the particles to begin and end in vacuum. The total fluence contained by the pulse is proportional to

$$\mathcal{E} = \int_0^{2\pi N} d\phi a^2(\phi) = \frac{3\pi}{8} N a_0^2. \quad (12)$$

In this work, \mathcal{E} is kept constant, which fixes a_0 for each value of N . It has been shown [23, 25] that the classical Landau–Lifshitz prediction for the final state of a particle distribution emerging from the pulse is completely determined by the fluence, whereas quantum effects are expected to depend on the value of a_0 itself. We are then able to explore the impact of the reduced emission in the quantum model with varying a_0 while maintaining the same classical prediction. This allows us to comment on the *relative* importance of the quantum effects.

To motivate this study, parameters have been chosen to be comparable to those predicted for the forthcoming ELI facility. We have chosen to consider $N a_0^2 = 4.624 \times 10^4$ which, for $N = 10$ with a wavelength of $\lambda = 800$ nm, represents a 27 fs pulse duration [26] with peak intensity 1×10^{22} W/cm². We have investigated collisions between pulses of length $N \in [2, 200]$ cycles (together with their corresponding a_0) and a bunch of $N_P = 401$ particles, with an initial momentum spread of 20% around $\bar{\gamma} = \sqrt{1 + \bar{p}^2} = 2 \times 10^3$. This corresponds to an average particle energy of just over 1 GeV, which should be within the capabilities of the linear accelerators at ELI.

Figure 1 shows the variation of the particle distribution function on the (ϕ, p) phase space. As can clearly be seen in moving from the classical Landau–Lifshitz theory (left) to the quantum model (right), there are noticeable differences in the mean \bar{p} , spread $\hat{\sigma}$, and skewness S of the distribution. We first note that the difference in measuring the initial value $\hat{\sigma}_i = 19.9\% < 20\%$ is due to the finite number of particles used to represent the distribution. Essentially, it comes down to the ‘ \simeq ’ in equation (6), compared to the definition given by equation (8). As N_P is increased, the approximation in equation (6) improves, and the measured value approaches the desired value. The value $N_P = 401$ was chosen as it allows the

error in the initial spread to be less than 0.5% (see Supplemental Material). In practice, good agreement can be found with lower N_P , with the caveat that properties sensitive to the tails of the distribution (such as skewness) may be strongly affected.

For the classical theory, the final distribution only depends on the fluence contained by the pulse, though this does not prevent the system from taking different routes along the way. As we decrease the number of cycles we observe in Fig. 1 very different intermediate behaviour, yet the measured properties of the final distribution support this prediction: in each case, we measure the mean momentum $\bar{p}_f = 465$ with a relative spread $\hat{\sigma}_f = 5.19\%$. This represents a significant contraction of the phase space, where the average energy of the particle bunch has significantly decreased, as has its thermal spread (beam cooling), and the distribution has become more sharply peaked. In addition, we find the development of a negatively-skewed distribution with $S_f = -1.0$. In the classical model, the higher a particle’s momentum the more it radiates. This causes particles in the positive tail of the distribution to be damped and slowed down more than those in the negative tail. We note that the classical result has been verified by comparison with the analytic solution of the Vlasov equation including the Landau–Lifshitz description of radiation reaction [23].

The introduction of a semi-classical model in which the effect of radiation reaction is reduced by the function $g(\chi)$ given by equation (4) causes a reduction in the phase space contraction. Figure 1(a) and (b) for $N = 200$ clearly indicates this point, with the final average momentum $\bar{p}_f = 574$ slightly higher than the classical case. The final relative momentum spread is now 8.24%, showing that the final distribution is no longer quite so sharply peaked. While remaining negative, the skewness has also reduced in magnitude to -0.58 , since it is precisely the higher-energy particles which were previously most affected by radiation reaction that now have this damping suppressed by larger χ (smaller $g(\chi)$).

These changes become even more pronounced as we move to higher intensities (by reducing the number of cycles). For $N = 10$, as shown in Fig. 1(c) and (d), we find that $\bar{p}_f = 904$ and $\hat{\sigma}_f = 14.5\%$ have both increased, with the skewness also increasing to $S_f = -0.16$. This trend continues to $N = 2$ as shown in Fig. 1(e) and (f). In this case, there is very little beam cooling for the quantum model, with the final relative momentum spread taking the value $\hat{\sigma}_f = 17.8\%$ around $\bar{p}_f = 1232$. The profile also remains much more Gaussian, with $S_f = -0.041$.

The reduction in phase space contraction observed here is in agreement with previous predictions [25]. This has allowed us to verify our method for reconstructing the particle distribution function and enabled the effects of the semi-classical model on a *distribution* of particles to be investigated. The distributions have been nicely reconstructed and do not feature any artefacts or lack of

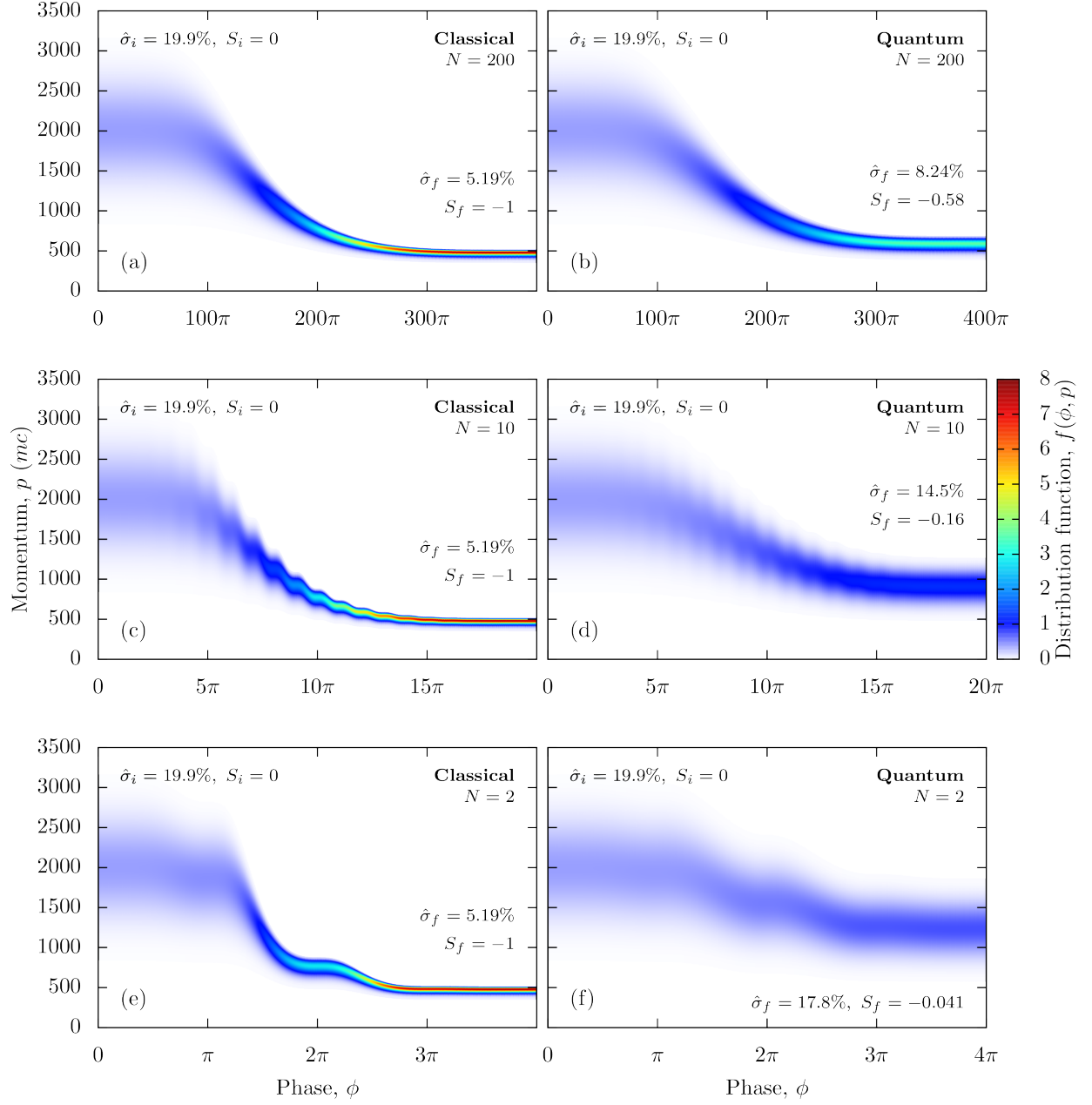


FIG. 1. (Color online) The phase space evolution of the distribution function $f(\phi, p)$. Classical predictions are shown in parts (a), (c) and (e), while the corresponding semi-classical results are presented in parts (b), (d) and (f). Values of the initial and final relative momentum spread and momentum skewness are displayed in each figure. The pulse length is reduced from $N = 200$ cycles in (a) and (b), to $N = 10$ cycles in (c) and (d), and finally to $N = 2$ cycles in (e) and (f). In each case, we observe an increase in the final mean momentum and its spread predicted by the quantum model.

resolution, unlike some other approaches [14].

Figure 1 also shows how the difference between the classical and quantum results increases as we increase the intensity (or decrease N). It is therefore interesting to consider the difference $\delta\hat{\sigma}_f = \hat{\sigma}_f^{\text{qm}} - \hat{\sigma}_f^{\text{cl}}$ as a fraction of the total (constant) classical change in momentum spread, $\Delta\hat{\sigma}^{\text{cl}} = \hat{\sigma}_i - \hat{\sigma}_f^{\text{cl}}$. This can be found in Fig. 2(a), where we see that for $N = 2$ the two predictions differ by around

85%. As N is increased, this ratio is reduced as the average quantum parameter $\langle\chi\rangle$ takes smaller values and radiation reaction is not so heavily suppressed. It would be expected that the two models converge as $N \rightarrow \infty$.

In cases where the variation between the predictions of the two theories are large, it is important to be confident that the model remains valid. As a semi-classical model, we expect it to remain valid into the weakly quantum

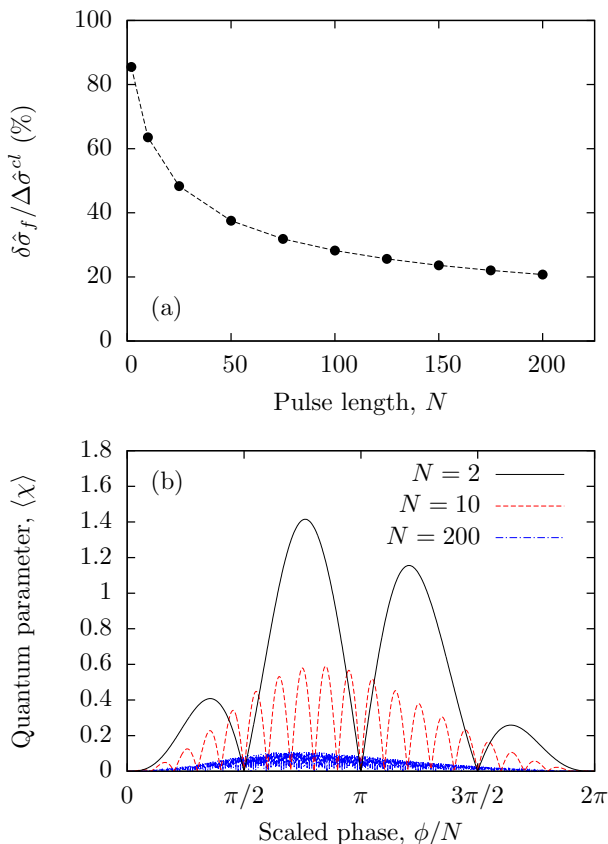


FIG. 2. Part (a): variation of the final relative momentum spread difference $\delta\hat{\sigma}_f$ as a percentage of the total classical change, $\Delta\hat{\sigma}^{cl}$. The evolution of the average quantum parameter $\langle\chi\rangle$ is plotted in part (b) with phase scaled by N in order to overlay the results. For $N \geq 10$, we find $\langle\chi\rangle < 1$ such that the semi-classical model should be valid.

regime, such that particles experience instantaneous values up to $\chi \sim O(1)$. Figure 2(b) shows the evolution of $\langle\chi\rangle$ as the bunch moves through the laser pulse. The values plotted are for the quantum model, since they remain smaller in the classical model. The cases $N = 10$ and 200 clearly remain below this threshold. The peak value $\langle\chi\rangle \sim 1.4$ observed for $N = 2$ is perhaps on the large side, but for such a short pulse the plane wave approximation may be considered questionable anyway.

In this Letter, we have introduced a novel method to efficiently and accurately calculate the distribution function for an electron beam interacting with an intense laser pulse. Using this method, we have compared classical and quantum predictions of radiative cooling, finding that quantum effects can significantly alter the beam properties, and unlike the classical case can be sensitive to the shape of the laser pulse.

The results presented in this Letter are limited to the semi-classical case $\chi \lesssim 1$. However, it should be noted that this restriction is due to the use of a deterministic

equation of motion, and not the method of sampling and reconstructing the distribution. There should be no obstruction to using this approach with a stochastic equation with photon emission probabilities determined by strong field QED, as in [27], to explore strongly quantum regimes. This will be addressed in future work.

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